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Preferential urn model and nongrowing complex networks

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A preferential urn model, which is based on the concept "the rich get richer," is proposed. From a relationship between a nongrowing model for complex networks and the preferential urn model in regard to degree distributions, it is revealed that a fitness parameter in the nongrowing model is interpreted as an inverse local temperature in the preferential urn model. Furthermore, it is clarified that the preferential urn model with randomness generates a fat-tailed occupation distribution; the concept of the local temperature enables us to understand the fat-tailed occupation distribution intuitively. Since the preferential urn model is a simple stochastic model, it can be applied to research on not only the nongrowing complex networks, but also many other fields such as econophysics and social sciences.

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Recently, complex networks have attracted a lot of interest in research fields of statistical physics [1]. In the research fields, many scientists have mainly focused their attention on growing networks in which a new node is added to networks with time. Barabási and Albert [2] have proposed an important growing model, so-called the Barabási and Albert (BA) model, and it has been revealed that scale-free networks with the degree distribution $P(k) \sim k^{-3}$ are generated by using the concepts of *growth* and *preferential attachment*. The concept of "preference" seems reasonable for explaining various phenomena in social science literature; the concept indicates the fact "the rich get richer."

While it has been revealed that the concept of preference could be sufficient to generate a fat-tailed degree distribution in the growing case, it is an open question whether the concept of preference is sufficient in the case of nongrowing networks. It has been shown that the concept of preference alone does not give networks with the fat-tailed degree distribution in the nongrowing case [1,3,4]. Though fat-tailed degree distributions are obtained for a wide range of parameters in the growing case, Dorogovtsev *et al.* [3] have shown that such fat-tailed degree distributions may exist only at a certain critical point in the nongrowing case. As for nongrowing networks, threshold models have succeeded in generating scale-free networks [5–7]. In the threshold models, intrinsic weights are different from each other, and the randomness is essential. Hence, it is expected that randomness plays an important role in generating the fat-tailed degree distribution in nongrowing cases.

While the threshold models make networks without dynamical processes, networks in the real world could have some dynamics, so that it is reasonable to consider dynamical models for nongrowing networks. One of the examples of dynamical processes is a rewiring process proposed by Albert and Barabási [8], though the rewiring process has been introduced to a growing model. For the case of nongrowing models, simple dynamics with the preferential concept do not generate a network with the fat-tailed degree distribution, as described above. However, a dynamical model with preferential rewiring processes and randomness has recently been proposed [4], and it has been revealed that the randomness in regard to fitness parameters could generate networks with a fat-tailed degree distribution.

In the present paper, we propose a new stochastic model in order to discuss and interpret nongrowing complex networks. The proposed stochastic model is based on urn models, which is widely used in physics, mathematics, economics, and so on. Several urn models have been studied in recent years [9–15]. The urn model proposed in the present paper is based on the preferential concepts, so that we refer to the new urn model as the *preferential urn model*. We show how the nongrowing model for complex networks relates to the preferential urn model, and the fitness parameters of the nongrowing model directly correspond to inverse local temperatures of urns. These interpretations and discussions allow us to understand the nongrowing networks more intuitively: nodes with high temperature tend to release edges attached to them, and those with low temperature do not. Furthermore, it is clarified that the preferential urn model with randomness generates a fat-tailed occupation distribution.

At first, we introduce a nongrowing model with the preferential concepts. We here consider an undirected network without growth. Starting a random network with N nodes and M edges, we repeat the following rewiring procedures.

(i) Select an edge l_{ij} at random.

(ii) Replace the edge l_{ij} by an edge l_{im}, where node m is chosen randomly with a probability

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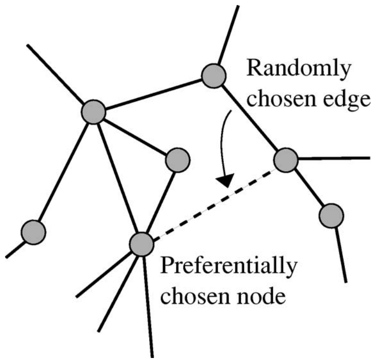

FIG. 1. Process of the rewiring. An edge is chosen *at random* and reconnected to a node selected preferentially.

$$\Pi_m \propto (k_m + 1)^{\beta_m}, \tag{1}$$

where k_m is the degree of node m, and β_m a fitness parameter of node m. The fitness parameters $\{\beta_i\}$ represent that each node has a different ability to compete for edges. A value of the fitness parameter β_i is chosen from a fitness distribution $\phi(\beta)$, and the fitness parameters are time independent.

The procedure is illustrated in Fig. 1.

We denote the occupation probability of nodes with degree k and fitness parameter β in $[\beta, \beta + d\beta]$ as $f_k(\beta,t)$. Using a rate equation approach, a time evolution of $f_k(\beta,t)$ is described as follows:

$$\frac{\partial f_k(\beta,t)}{\partial t} = -\frac{(k+1)^{\beta} f_k(\beta,t)}{Z(t)} + \frac{k^{\beta} f_{k-1}(\beta,t)}{Z(t)} - \frac{k f_k(\beta,t)}{M} + \frac{(k+1) f_{k+1}(\beta,t)}{M}, \tag{2}$$

where $Z(t)$ is the normalization constant defined by $Z(t) = \int d\beta \Sigma_k (k+1)^{\beta} f_k(\beta,t)$, and M is the number of edges in the whole network. The similar approach to nongrowing models with fitness parameters is described in Ref. [4]. However, the analytical treatments using the rate equation have not succeeded in calculating the degree distribution in the case of the models with randomness because of their complexity.

Next, we introduce the preferential urn model (Fig. 2). Details of the analysis of general urn models are briefly reviewed in Ref. [12]. While the definition of energy in the preferential urn model is different from previous urn models, general formalisms for analysis are similar. We here consider the system in which there are N urns and M balls. The number of balls in urn i is denoted as n_i, and then $M = \Sigma_{i=1}^{N} n_i$. An energy of each urn is defined by

$$E(n_i) = -\ln(n_i!), \tag{3}$$

and hence the Hamiltonian of the whole system is $\mathcal{H} = \Sigma_{i=1}^{N} E(n_i)$. Using the energy, we calculate the unnormalized Boltzmann weight attached to urn i as follows:

$$p_{n_i} = \exp[-\beta_i E(n_i)] = (n_i!)^{\beta_i}. \tag{4}$$

For general use, we assume that each inverse temperature β_i could have a different value. In physics, it might seem strange that components of the system have different temperatures, but in information physics or social sciences, there should be no restriction for the inverse temperatures (or one may say that the system is nonequilibrium). We denote the inverse temperature as *the inverse local temperature* because of its locality. The unnormalized Boltzmann weight allows us to determine dynamics of the urn model. Using the heat-bath rule, we calculate the transition rate $W_{n_i \to n_i+1}$ from the state n_i to n_i+1 as follows [16]:

$$W_{n_i \to n_i+1} \propto \frac{p_{n_i+1}}{p_{n_i}} = \frac{\{(n_i+1)!\}^{\beta_i}}{\{(n_i)!\}^{\beta_i}} = (n_i+1)^{\beta_i}. \tag{5}$$

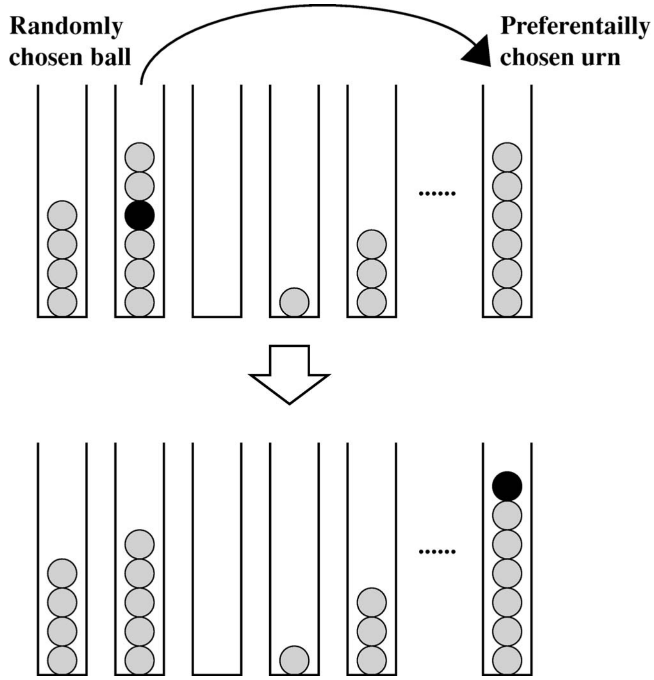

FIG. 2. Preferential urn model. A ball is chosen at random, and the ball is moved to an urn selected preferentially.

Therefore, the dynamics of the preferential urn model is summarized as follows:

(1) Choose a ball randomly.

(2) Move the drawn ball to an urn selected by the transition rate $W_{n_i \to n_i+1}$.

It should be noted that the dynamics with the above transition rate, $W_{n_i \to n_i+1}$, lead to the same rate equation of Eq. (2). Therefore, we consider the preferential urn model instead of the nongrowing model in regard to the degree distribution; we regard the number of balls in each urn as that of edges attached to each node.

Interpreting the nongrowing complex networks as the preferential urn model, we can understand the nongrowing complex networks more intuitively. First, each node tends to get edges more and more because the energy of Eq. (3) shows each node is stable when the node obtains a lot of edges. Additionally, each node competes for edges because of the restriction that the total number of edges is fixed. Second, the fitness parameter β_i directly corresponds to the inverse local temperature. Considering the fitness parameter

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as the local temperature, we easily lead to the following interpretation: nodes with higher temperature are likely to kick out edges, and those with lower temperature tend to store edges.

Next, we analytically calculate the occupation probability of the preferential urn model. To calculate the occupation probability, the partition function of the preferential urn model is considered. We denote the average density (average degree) as ρ, and then $M=\rho N$. The partition function is calculated as follows [12]:

$$\begin{aligned} Z(N,M) &= \sum_{n_1=0}^{\infty}\cdots\sum_{n_N=0}^{\infty}\frac{p_{n_1}}{n_1!}\cdots\frac{p_{n_N}}{n_N!}\delta\left(\sum_{i=1}^{N}n_i,M\right) \\ &= \sum_{n_1=0}^{\infty}\cdots\sum_{n_N=0}^{\infty}\frac{p_{n_1}}{n_1!}\cdots\frac{p_{n_N}}{n_N!}\oint\frac{dz}{2\pi i}z^{\Sigma_i n_i-M-1} \\ &= \oint\frac{dz}{2\pi i}\frac{1}{z^{M+1}}[G(z)]^N \\ &= \oint\frac{dz}{2\pi i}\exp\left\{-\ln z+M\left(\ln z+\frac{1}{\rho}\ln G(z)\right)\right\}, \end{aligned} \tag{6}$$

where

$$G(z)=\sum_k\frac{p_k}{k!}z^k. \tag{7}$$

We calculate the partition function using the saddle-point method, and the occupation probability of urn 1 is obtained by

$$\begin{aligned} f_{k,\mathrm{eq}} &\equiv \frac{1}{Z(N,M)}\sum_{n_1=0}^{\infty}\cdots\sum_{n_N=0}^{\infty}\delta(n_1,k)\frac{p_{n_1}}{n_1!}\cdots\frac{p_{n_N}}{n_N!}\delta\left(\sum_{i=1}^{N}n_i,M\right) \\ &= \frac{p_k}{k!}\frac{z_s^k}{G(z_s)}, \end{aligned} \tag{8}$$

where z_s is the saddle-point value

$$\rho=\frac{z_sG'(z_s)}{G(z_s)}. \tag{9}$$

In what follows, we explain three examples of the preferential urn models; two of them are homogeneous models in which all the inverse local temperatures have the same values; the other one has randomness in the inverse local temperatures.

(1) *High-temperature limit*. We consider the case $\beta_i\to 0$ for all i. Therefore, the distribution of the inverse temperature is $\phi(\beta)=\delta(\beta)$. In this case, it is easy to calculate the degree distribution from Eqs. (4), (8), and (9), and then the occupation probability follows a Poisson distribution

$$P(k)=f_{k,\mathrm{eq}}=e^{-\rho}\frac{\rho^k}{k!}. \tag{10}$$

The occupation probability corresponds to that of ordinary random networks [1].

(2) *The case without randomness*. When all urns have the same inverse temperature, i.e., $\beta_i=1$ for all i, we obtain the following occupation probability from Eqs. (4), (8), and (9):

$$P(k)=f_{k,\mathrm{eq}}=\frac{1}{1+\rho}\exp\left\{-k\ln\left(1+\frac{1}{\rho}\right)\right\}. \tag{11}$$

The occupation probability of Eq. (11) is the same one in Ref. [4].

(3) *The case with randomness*. When there is randomness in the inverse local temperatures, we can also perform the similar analytical treatments. Generally speaking, it is difficult to analyze the model with randomness, and the analytical treatments become very complicated. In the analysis of the preferential urn model with randomness, we use the replica analysis [17]. In the analysis, we assume the self-averaging property of the equilibrium occupation probability

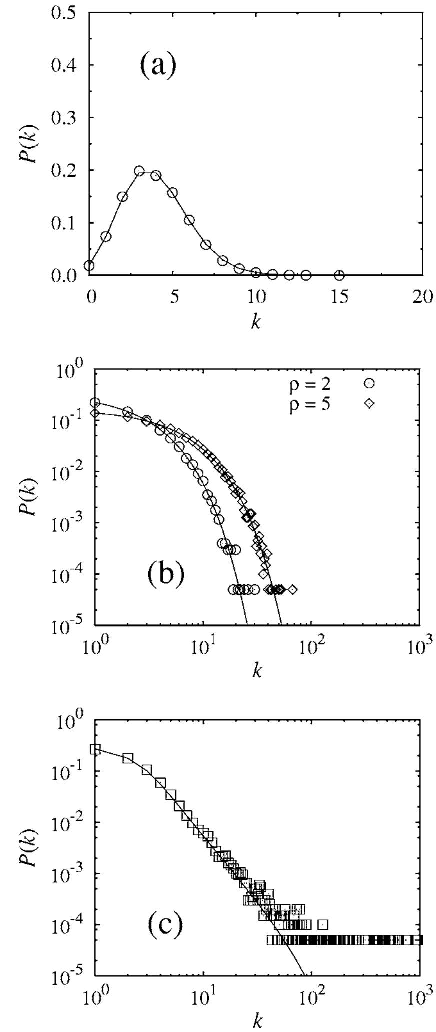

FIG. 3. Examples of the occupation distributions of the preferential urn model. (a) The case of high-temperature limit $[\phi(\beta)=\delta(\beta)]$. The density is $\rho=4.0$. (b) The case with nonrandomness $[\phi(\beta)=\delta(\beta-1)]$. (c) The case with uniform randomness $(\phi(\beta)=1,(\beta\in[0,1]))$. The density is $\rho=4.0$. In all the cases of (a), (b), and (c), the number of urns is $N=1000$ and $2000N$ ball-exchange processes have been performed. The data are averaged over 20 samplings.

and replica symmetry. The analytical treatment [18] leads to the following occupation probability in the case of the uniform distribution of the inverse local temperatures, $\phi(\beta)=1$, $\beta\in[0,1]$:

$$P(k) = \int_0^1 \frac{(k!)^{\beta-1} z_s^k}{\sum_{n=0}^{\infty} (n!)^{\beta-1} z_s^n} d\beta. \quad (12)$$

When the density is sufficiently large, i.e., $\rho \gg 1$, we obtain the approximate form of the occupation probability as $P(k) \sim k^{-2}(\ln k)^{-2}$ [18]; the occupation distribution follows a generalized power law with a squared inverse logarithmic correction.

These analytical treatments make clear that the concept of local temperatures is useful to understand the fat-tailed occupation probability in the case with randomness. When there is no randomness, all urns compete for balls, and hence a fat-tailed behavior does not occur; in the case with randomness, several urns with low temperature tend to store more balls than those with high temperature, and hence urns with many balls could emerge. Therefore, using the concept of the local temperatures, we can intuitively understand the fat-tailed occupation probability.

Finally, we give comparisons between the results of the analytical treatments and numerical experiments. Figure 3 shows the comparisons. In the numerical experiments, we use the preferential urn model with the number of urns $N=1000$. Initially, each inverse local temperature is determined by using a distribution $\phi(\beta)$, and balls are distributed randomly. Next, we repeat the ball-exchange procedures; select a ball randomly, and move the drawn ball to an urn selected by using $W_{n_i \to n_i+1}$ of Eq. (5). The number of the ball-exchange procedures are $2000N$, and we checked that this number of exchange procedures is enough to give equilibrium states. In Fig. 3(a), we set the density as $\rho=4.0$, and the distribution of the inverse local temperatures as $\phi(\beta)=\delta(\beta)$. The solid line in Fig. 3(a) corresponds to Eq. (10). Figure 3(b) shows the results of the case $\phi(\beta)=\delta(\beta-1)$. We have calculated two cases with the density $\rho=2.0$ and $\rho=5.0$, and the solid lines in Fig. 3(b) correspond to Eq. (11) with the density $\rho=2.0$ and $\rho=5.0$, respectively. In Fig. 3(c), we show the result of the case $\phi(\beta)=1$, $\beta\in[0,1]$. We set the density as $\rho=4.0$, and the solid line in Fig. 3(c) corresponds to Eq. (12) with $z_s=0.97$. Those results are in good agreement with corresponding analytical results.

In summary, we propose a different urn model. From analysis of the preferential urn model, it has been revealed that there is a direct relationship between the nongrowing model with preferential rewiring processes and the preferential urn model in regard to the degree distribution. The relationship allows us to introduce the concepts of energy and the inverse local temperature to the nongrowing complex networks, and it has been shown that each fitness parameter is directly interpreted as each inverse local temperature. These concepts let us understand the nongrowing complex networks intuitively. We note that the preferential urn model would be widely applicable. The concept "the rich get richer" would be important for various systems such as econophysics and social sciences. Without the growing property, fat-tailed distributions are obtained by the concept of randomness; it is likely that each agent (or component) of a system has different ability and this corresponds to the randomness of the inverse local temperatures.

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